

SCHUR MULTIPLIERS OF NILPOTENT LIE ALGEBRAS

LINDSEY R. BOSKO lrbosko@ncsu.edu

ERNIE L. STITZINGER stitz@ncsu.edu

*Mathematics Department, North Carolina State University
Raleigh, North Carolina 27695, United States of America*

ABSTRACT. We consider the Schur multipliers of finite dimensional nilpotent Lie algebras. If the algebra has dimension greater than one, then the Schur multiplier is non-zero. We give a direct proof of an upper bound for the dimension of the Schur multiplier as a function of class and the minimum number of generators of the algebra. We then compare this bound with another known bound.

Keywords: Schur multiplier

1. Introduction

The Lie algebra analogue of the Schur multiplier was investigated in the dissertations of Kay Moneyhun and Peggy Batten (see [8] and [2]). Among their results is that if $\dim L = n$, then $\dim M(L) \leq \frac{1}{2}n(n-1)$ where $M(L)$ is the Schur multiplier of the Lie algebra, L . A number of other results bounding $\dim M(L)$ have appeared (see [3], [5], [9], [10], and [13]). In this note several other bounds are provided. In particular, we show that if L is nilpotent and $\dim L > 1$, then $\dim M(L) \neq 0$. This result can be contrasted with a result of Johnson in [6] which shows that a p -group with trivial multiplier has restrictions placed on it. We also find an upper bound for $\dim M(L)$ as a function of class and the number of generators for L . This result is similar to Theorem 3.2.5 of [7]. This Lie algebra result follows as a consequence of a general theory, provided in [10] but we give a short, direct proof and contrast this bound with one found in [5].

2. Preliminaries

Suppose that L is generated by n elements. Let F be a free Lie algebra generated by n elements and $L \cong F/R$. Since R is an ideal in

F , R is also free. Witt's formula from [1] gives us

$$\dim F^d/F^{d+1} = \frac{1}{d} \sum_{m|d} \mu(m) n^{d/m} \equiv l_n(d) \quad (2.1)$$

where μ is the Möbius function. Hence, F/F^t is finite dimensional and nilpotent for all t .

Let N be an ideal in L and S be an ideal in F such that $(S+R)/R \cong N$. Recall that $M(L) = (F^2 \cap R)/[F, R]$ ([2]). Then $M(L/N) \cong (F^2 \cap (S+R))/[F, S+R]$. It is routine to verify that there is a natural exact sequence

$$0 \rightarrow \frac{R \cap [F, S]}{[F, R] \cap [F, S]} \rightarrow M(L) \rightarrow M(L/N) \rightarrow \frac{N \cap L^2}{[N, L]} \rightarrow 0. \quad (2.2)$$

Note that covers and multipliers can be computed using the GAP program [4].

3. Trivial $M(L)$

It is shown in [6] that if G is a finite p -group with $M(G) = e$, then severe restrictions are placed on G . For further work in the problem see [11] and also [12] for a simpler proof. We will show that if L is a nilpotent Lie algebra with $M(L) = 0$ then $\dim L \leq 1$.

Let L be a nilpotent Lie algebra generated by $n > 1$ elements. Hence, $\dim L/L^2 = n$. Let F be a free Lie algebra generated by n elements with $L \cong F/R$. Suppose that L has class c . Hence, $F^{c+2} \subsetneq F^{c+1} \subseteq R$ using the result in the last section. Furthermore, F/F^{c+2} is finite dimensional and nilpotent of class $c+1$. Then,

$$n = \dim L/L^2 = \dim \frac{F/R}{(F/R)^2} = \dim \frac{F}{F^2 + R} \leq \dim F/F^2 = n.$$

Lemma 3.1. *If L is nilpotent and $\dim L = n > 1$, then $R \subseteq F^2$ and $M(L) \cong R/[F, R]$.*

Theorem 3.2. *If L is a finite dimensional nilpotent Lie algebra of dimension greater than 1 and class c , then $M(L) \neq 0$.*

Proof. Continuing with the notation, $L \cong F/R$, $F^{c+2} \subsetneq F^{c+1} \subseteq R$, and F/F^{c+2} is nilpotent. Hence, $[F, R] \subsetneq R$ and $M(L) \cong R/[F, R] \neq 0$. \square

4. An Upper Bound for $\dim M(L)$

Suppose L has class $c \geq 2$. Let $N = L^c$ and $S = F^c$ in Eq. 2.2. Then $L^c \cong (F^c + R)/R$ and $[F, S] = F^{c+1} \subseteq R$ since $L^{c+1} = 0$. Hence,

Eq. 2.2 becomes

$$0 \rightarrow \frac{F^{c+1}}{[F, R] \cap F^{c+1}} \xrightarrow{\sigma} M(L) \rightarrow M(L/L^c) \rightarrow L^c \rightarrow 0. \quad (4.1)$$

Theorem 4.1. *Let L be a nilpotent Lie algebra of class c which is generated by n elements. Then*

$$\dim M(L) \leq \sum_{j=1}^c l_n(j+1)$$

where $l_n(q) = \frac{1}{q} \sum_{s|q} \mu(s) n^{q/s}$.

Proof. Induct on c . Let F be free of rank n where $L \cong F/R$. If $c = 1$, then F/R is abelian, $F^2 \subseteq R$ and $M(L) = F^2/[F, R]$. Since $F^3 \subseteq [F, R]$, $\dim M(L) \leq \dim F^2/F^3 = l_n(2)$. Now, suppose that $c > 1$. By induction, $\dim M(L/L^c) \leq t \equiv \sum_{j=1}^{c-1} l_n(j+1)$. In Eq. 4.1, let $A = \text{Im}(\sigma)$. Then $M(L)/A \cong B \subseteq M(L/L^c)$. Hence, $\dim M(L)/A \leq t$. But $F^{c+1} \subseteq R$ and $F^{c+2} \subseteq [F, R] \cap F^{c+1}$. Thus, A is the homomorphic image of F^{c+1}/F^{c+2} . Therefore $\dim A \leq l_n(c+1)$ by Eq. 2.1 and $\dim M(L) \leq t + l_n(c+1)$ as desired. \square

We compare our result to the upper bound given in [5]:

Theorem 4.2. *If L is a Lie algebra of dimension n , then*

$$\dim M(L) \leq \frac{1}{2}n(n-1) - \dim L^2.$$

We now examine the two theorems applied to different Lie algebras.

Example 4.3. Let F be a free Lie algebra on 2 generators and $L = F/F^3$. Then L is a Lie algebra of 2 generators and class 2. So, $L \supseteq L^2 \supseteq L^3 = 0$. Then, $\dim L/L^2 = l_2(1) = 2$, $\dim L^2/L^3 = l_2(2) = \frac{1}{2}[\mu(1)2^2 + \mu(2)2] = \frac{1}{2}(4-2) = 1$. Thus, $\dim L = 3$ and by Theorem 4.2, $\dim M(L) \leq 2$. By Theorem 4.1,

$$\begin{aligned} \dim M(L) &\leq \sum_{j=1}^2 l_2(j+1) = l_2(2) + l_2(3) \\ &= 1 + \frac{1}{3}(\mu(1)2^3 + \mu(3)2) \\ &= 1 + \frac{1}{3}(6) = 3. \end{aligned}$$

Thus, the result of Theorem 4.2 proves to be a better bound than the one obtained by our new theorem.

Example 4.4. Let F be a free Lie algebra of 2 generators and $L = F/F^4$. Then L is a Lie algebra of 2 generators and class 3. Thus, $\dim L = 5$ and by Hardy's Theorem, $\dim M(L) \leq 7$. By LB's Theorem,

$$\begin{aligned}
 \dim M(L) &\leq \sum_{j=1}^3 l_2(j+1) = l_2(2) + l_2(3) + l_2(4) \\
 &= 1 + 2 + \frac{1}{4}[\mu(1)2^4 + \mu(2)2^2 + \mu(4)2] \\
 &= 3 + \frac{1}{4}(16 - 4) \\
 &= 3 + 3 = 6
 \end{aligned}$$

Thus, we see that in this case, our theorem creates a better upper bound for $\dim M(L)$ than the previously known result.

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